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The Formulation of Quantum Mechanics in Terms of Nuclear Algebras

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In this work we analyze the convenience of nuclear barreled b*-algebras as a better mathematical framework for the formulation of quantum principles than the usual algebraic formalism in terms of C*-algebras. Unbounded operators on Hilbert spaces have an abstract counterpart in our approach. The main results of the C*-algebra theory remain valid. We demonstrate an extremal decomposition theorem, an adequate functional representation theorem, and an extension of the classical GNS theorem.

1. INTRODUCTION

In the framework of the formulation of quantum mechanics in terms of "abstract" C*-algebras, observable magnitudes are represented by the Hermitian elements of the algebra that characterizes the physical system under consideration.

The GNS construction leads to a representation of this algebra as an algebra of bounded operators on a Hilbert space [Bratteli]; unbounded operators do not have an abstract counterpart in this algebraic approach.

This fact is really a disadvantage of the theory: the algebraic approach is intended to be a generalization of the traditional Hilbert space formulation of quantum mechanics and in this context most of the generators of the symmetry groups that appear—in particular, most of the Hamiltonian operators, the momentum operator, and the position operator—are given by unbounded operators, and then, in algebraic terms, they cannot represent measurable quantities.

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The problem could be solved in several ways (see, for example, [Ruelle] and [Roberts]), but the most natural seems to be the consideration of topological algebras not necessarily normed.

Obviously, these algebras must satisfy certain conditions in order to guarantee the physical consistency of the theory.

In this work we study the case of complete locally multiplicatively convex *-algebras (that is, complete locally convex *-algebras in which the topology is defined by a basis of neighborhoods of zero composed by idempotent sets) that, in addition, are symmetric, barreled, and nuclear.

Complete locally multiplicatively convex symmetric *-algebras are the natural generalization of C*-algebras. Commonly they are called *b*-algebras* [Allan]. The nuclearity condition guarantees the validity of some of the properties of finite-dimensional algebras and regularizes the tensor product [Treves]. Finally, we ask for barreledness in order to have a "nice" spectral theory.

We will show that the main results of the formulation in terms of C*-algebras remain valid. For example, we will demonstrate an extremal decomposition theorem analogous to the classical Krein–Millman lemma (see Section 3). We will prove also that every closed commutative subalgebra admits a functional representation on a locally compact space in such a way that the topology coincides with the topology of uniform convergence on its compact subsets (see Section 4). On the other hand, we will show that the generalization of the GNS representation leads to an essentially self-adjoint representation of the algebra as an algebra of (in general, unbounded) operators on a separable Hilbert space with a common domain that, provided with a convenient topology, constitutes a rigged Hilbert space (see Section 5).

As examples of nuclear barreled b*-algebras that are commonly used in physics we mention the tensor algebra over a nuclear Fréchet space (provided, of course, with a conjugation) and the trivial case of finite-dimensional C*-algebras (see Section 6).

2. b*-ALGEBRAS: DEFINITIONS AND GENERALITIES

Definition 1. Let $\mathscr{A}$ be a locally convex algebra. $\mathscr{A}$ is a locally multiplicatively convex algebra if and only if there exists a basis of neighborhoods of zero entirely composed of convex idempotent sets, i.e., convex sets that contain their squares.

In every locally convex space the topology can be defined by a basis of continuous seminorms. In addition, in the case of a locally multiplicatively convex algebra, this basis can be chosen in such a way that each seminorm is a submultiplicative one.

Proposition 1 [Michael][3].

1. Every normed algebra is locally multiplicatively convex.
2. The direct product of a collection of locally multiplicatively convex algebras is also a locally multiplicatively convex algebra.
3. Every subalgebra of a locally multiplicatively convex algebra is locally multiplicatively convex in the relative topology.
4. The quotient algebra of a locally multiplicatively convex algebra modulo any of its closed bilateral ideals is locally multiplicatively convex in the inherited topology.
5. An algebra with the weaker topology that makes continuous every homomorphism included in a separating family of homomorphisms from the algebra in a corresponding collection of locally multiplicatively convex algebras is a locally multiplicatively convex algebra.

Proposition 2 [Arens]. In every locally multiplicatively convex algebra the multiplication law is jointly continuous. Moreover, if the algebra has a unit, then inversion is continuous on the subalgebra of invertible elements.

Locally multiplicatively convex algebras are closely related to normed algebras, as is shown by the following theorem.

Theorem 1. A topological algebra $\mathcal{A}$ is locally multiplicatively convex if and only if it is isomorphic to a subalgebra of the direct product of a collection of normed algebras. Moreover, if $\mathcal{A}$ is complete, then it is locally multiplicatively convex if and only if it is the Hausdorff projective limit of a family of Banach algebras.

Proof. First, let us assume that $\mathcal{A}$ is a locally multiplicatively convex algebra not necessarily complete and let $\{p_\alpha\}_{\alpha\in I}$ be a basis of submultiplicative seminorms defined on $\mathcal{A}$.

Clearly, the kernel of each of these seminorms

$$Ker(p_\alpha) = \{x \in \mathcal{A}: p_\alpha(x) = 0\} \tag{1}$$

is a bilateral closed ideal of $\mathcal{A}$, that is,

$$\mathcal{A} Ker(p_\alpha) = Ker(p_\alpha)\mathcal{A} = Ker(p_\alpha) \tag{2}$$

for every $\alpha \in I$.

Consider, then, for each $\alpha \in I$, the quotient algebra $\mathcal{A}/Ker(p_\alpha)$ and the respectively submultiplicative quotient norm

[3] In brackets we will indicate the references in which the demonstrations of the propositions and theorems can be found. We will reproduce just those demonstrations that are important for the comprehension of this work.

$$\tilde{p}_\alpha: \quad \mathscr{A}/Ker(p_\alpha) \to \mathbb{R}_+$$
$$\phi_\alpha(x) \to \tilde{p}_\alpha(\phi_\alpha(x)) = p_\alpha(x) \tag{3}$$

where

$$\phi_\alpha: \quad \mathscr{A} \to \mathscr{A}/Ker(p_\alpha)$$
$$x \to \phi_\alpha(x) \tag{4}$$

is the canonical homomorphism from $\mathscr{A}$ on $\mathscr{A}/Ker(p_\alpha)$.

Finally, let $\mathscr{A}_\alpha$ be the normed algebra which is $\mathscr{A}/Ker(p_\alpha)$ provided with the topology defined by $\tilde{p}_\alpha$.

The topology in $\mathscr{A}$ is the weaker that makes each ϕ_α a continuous application. So,

$$\phi: \quad \mathscr{A} \to \prod_{\alpha \in I} \mathscr{A}_\alpha$$
$$x \to \phi(x) = (\phi_\alpha(x))_{\alpha \in I} \tag{5}$$

is continuous. With $\{p_\alpha\}_{\alpha \in I}$ a basis of seminorms of $\mathscr{A}$, given $x \in \mathscr{A}$, $x \neq 0$, there exists an index $\alpha \in I$ such that $p_\alpha(x) = \tilde{p}_\alpha(\phi_\alpha(x)) \neq 0$, so, for each $x \in \mathscr{A}$, $x \neq 0$, one has that $\phi_\alpha(x) \neq 0$; then $\phi(x) \neq 0$ for every $x \in \mathscr{A}$, $x \neq 0$, and $\mathscr{A}$ is isomorphic to a subalgebra of $\prod_{\alpha \in I} \mathscr{A}_\alpha$.

The converse is immediate from Proposition 1, and then the first part of the theorem is proved.

Suppose now that $\mathscr{A}$ is complete and let $\tilde{\mathscr{A}}_\alpha$ be the Banach algebra which is the completion of $\mathscr{A}_\alpha$ for every $\alpha \in I$.

Making $\alpha \geq \beta$ if and only if $\mathscr{U}_\alpha \subset \mathscr{U}_\beta$, where $\mathscr{U}_\alpha$ and $\mathscr{U}_\beta$ are, respectively, the unitary closed balls of $\mathscr{A}_\alpha$ and $\mathscr{A}_\beta$, it is possible to order the index set I and make it into a directed system.

So, considering that for each pair $x, y \in \mathscr{A}$, $\phi_\alpha(x) = \phi_\alpha(y)$ implies that $\phi_\beta(x) = \phi_\beta(y)$, with $\alpha \geq \beta$, one can define in this case

$$\phi_{\alpha\beta}: \quad \mathscr{A}_\alpha \to \mathscr{A}_\beta$$
$$\phi_\alpha(x) \to \phi_{\alpha\beta}(\phi_\alpha(x)) = \phi_\beta(x) \tag{6}$$

and clearly one has that each of these mappings is a continuous homomorphism on $\mathscr{A}_\beta$. Then, each $\phi_{\alpha\beta}$ can be uniquely extended to a continuous homomorphism $\tilde{\phi}_{\alpha\beta}$ from $\tilde{\mathscr{A}}_\alpha$ into $\tilde{\mathscr{A}}_\beta$.

Consider now the direct product $\prod_{\alpha \in I} \tilde{\mathscr{A}}_\alpha$ and let P_α be the projector of $\prod_{\alpha \in I} \tilde{\mathscr{A}}_\alpha$ on $\tilde{\mathscr{A}}_\alpha$.

It is clear that the Hausdorff projective limit of the collection of Banach algebras $\{\tilde{\mathscr{A}}_\alpha\}_{\alpha \in I}$ corresponding to the collection of homomorphisms $\{\tilde{\phi}_{\alpha\beta}\}_{\alpha,\beta \in I}$, that is,

$$\mathcal{LP}(\tilde{\mathcal{A}}_\alpha, \tilde{\phi}_{\alpha\beta}) = \left\{ x \in \prod_{\alpha \in I} \tilde{\mathcal{A}}_\alpha \colon \tilde{\phi}_{\alpha\beta}(P_\alpha(x)) = P_\beta(x) \text{ when } \alpha \geq \beta \right\} \quad (7)$$

is a subalgebra of $\prod_{\alpha \in I} \tilde{\mathcal{A}}_\alpha$ and that $\{\mathcal{LP}(\tilde{\mathcal{A}}_\alpha, \tilde{\phi}_{\alpha\beta}) \cap P_\alpha^{-1}(\mathcal{U}_\alpha)\colon \alpha \in I\}$ is a basis of neighborhoods of zero for $\mathcal{LP}(\tilde{\mathcal{A}}_\alpha, \tilde{\phi}_{\alpha\beta})$.

Consider, again, the application ϕ defined by equation (5). This homomorphism defines an isomorphism from $\mathcal{A}$ into $\prod_{\alpha \in I} \tilde{\mathcal{A}}_\alpha$, an isomorphism that we will denote by $\tilde{\phi}$. Because $\tilde{\phi}(\mathcal{A}) \subset \mathcal{LP}(\tilde{\mathcal{A}}_\alpha, \tilde{\phi}_{\alpha\beta})$, one has that $\mathcal{A}$ is, in fact, isomorphic to a close subalgebra $\mathcal{LP}(\tilde{\mathcal{A}}_\alpha, \tilde{\phi}_{\alpha\beta})$ and one can identify it with its image under $\tilde{\phi}$ in $\mathcal{LP}(\tilde{\mathcal{A}}_\alpha, \tilde{\phi}_{\alpha\beta})$. With this convention is immediate that for every $\alpha \in I$ one has that $P_\alpha(x) = \tilde{\phi}_\alpha(x)$ for every $x \in \mathcal{A}$, where we are denoting by $\tilde{\phi}_\alpha$ the homomorphism that defines ϕ_α into $\tilde{\mathcal{A}}_\alpha$.

It will be sufficient, then, to show that $\tilde{\phi}(\mathcal{A})$ is dense in $\mathcal{LP}(\tilde{\mathcal{A}}_\alpha, \tilde{\phi}_{\alpha\beta})$. Now, $\{\mathcal{LP}(\tilde{\mathcal{A}}_\alpha, \tilde{\phi}_{\alpha\beta}) \cap P_\alpha^{-1}(\mathcal{U}_\alpha)\colon \alpha \in I\}$ is a basis of neighborhoods of zero for $\mathcal{LP}(\tilde{\mathcal{A}}_\alpha, \tilde{\phi}_{\alpha\beta})$, so it is enough to demonstrate that $P_\alpha(\tilde{\phi}(\mathcal{A}))$ is dense in $P_\alpha(\mathcal{LP}(\tilde{\mathcal{A}}_\alpha, \tilde{\phi}_{\alpha\beta}))$ for every $\alpha \in I$. But $P_\alpha(\tilde{\phi}(\mathcal{A})) = \tilde{\phi}_\alpha(\mathcal{A}) = \mathcal{A}_\alpha$, and, on the other hand, $P_\alpha(\mathcal{LP}(\tilde{\mathcal{A}}_\alpha, \tilde{\phi}_{\alpha\beta})) \subset \tilde{\mathcal{A}}_\alpha$. Considering that $\mathcal{A}_\alpha$ is dense in $\tilde{\mathcal{A}}_\alpha$, the statement is proved.

The demonstration of the converse follows immediately from the definition of $\mathcal{LP}(\tilde{\mathcal{A}}_\alpha, \tilde{\phi}_{\alpha\beta})$ and from Proposition 1. We omit the details. ■

Proposition 3 [Michael]. Let $\mathcal{A}$ be a locally multiplicatively convex algebra and $\{\mathcal{A}_\alpha\}_{\alpha \in I}$ as in the previous theorem.

1. $\mathcal{A}$ has a unit if and only if, for every $\alpha \in I$, $\mathcal{A}_\alpha$ has a unit.
2. An element $x \in \mathcal{A}$ is invertible if and only if, for every $\alpha \in I$, $\phi_\alpha(x) \in \mathcal{A}_\alpha$ is invertible.

Now, we turn our attention to the concept of a spectrum.

Definition 2. Given a topological algebra $\mathcal{A}$, an element $x \in \mathcal{A}$ is bounded if there exists a complex number λ, $\lambda \neq 0$, such that the set $\{(\lambda x)^n\colon n = 0, 1, 2, \ldots\}$ is bounded.

Definition 3. Let $\mathcal{A}$ be a topological algebra with unit and $x \in \mathcal{A}$ an arbitrary element. The resolvent set of x, $\rho(x)$, is the collection of complex numbers $\lambda \in \mathbb{C}$ such that $(\lambda \mathbf{1} - x)^{-1}$ exists and is bounded, with the point ∞ if x is not bounded. The spectrum of x, $\sigma(x)$, is the complement of the resolvent set of x, that is,

$$\sigma(x) = \rho(x)^c = \overline{\mathbb{C}} - \rho(x) \quad (8)$$

where $\overline{\mathbb{C}}$ is the one-point compactation of the complex plane. Finally, the spectral radius of x, $\Sigma(x)$, is given by

$$\Sigma(x) = sup\{|\lambda|: \lambda \in \sigma(x)\} \tag{9}$$

Remark 1. Observe that eventually it can occur that $\Sigma(x) = \infty$.

Proposition 4 [Arens]. The spectrum of an element of a locally convex algebra with unit and continuous inversion is a nonempty set.

Proposition 5 [Allan]. Let $\mathscr{A}$ be a locally convex algebra with unit and $x, y \in \mathscr{A}$.

1. If $P(x)$ is a polynomial in x and $\lambda \in \sigma(x)$, then $P(\lambda) \in \sigma(P(x))$.
2. If x is invertible and $\lambda \in \sigma(x)$, then $\lambda^{-1} \in \sigma(x^{-1})$.
3. If $\lambda \in \sigma(xy)$ and $\lambda \neq 0$, then $\lambda \in \sigma(yx)$.

Theorem 2. Let $\mathscr{A}$ be a complete locally multiplicatively convex algebra with unit and $\{\tilde{\mathscr{A}}_\alpha\}_{\alpha \in I}$ the collection of Banach algebras as in Theorem 1.

1. For every element $x \in \mathscr{A}$ one has that

$$\sigma(x) = \bigcup_{\alpha \in I} \sigma_\alpha(\tilde{\phi}_\alpha(x)) \tag{10}$$

   where $\tilde{\phi}_\alpha$ is the canonical homomorphism from $\mathscr{A}$ in $\tilde{\mathscr{A}}_\alpha$.
2. For every element $x \in \mathscr{A}$ the following is satisfied:

$$\Sigma(x) = \sup_{\alpha \in I} \Sigma_\alpha(\tilde{\phi}_\alpha(x)) = \sup_{\alpha \in I} \lim_{n \to \infty} \sqrt[n]{p_\alpha(x^n)} \tag{11}$$

Proof. 1. For every $\lambda \in \mathbb{C}$, $\lambda \neq 0$, one has that $\lambda \in \sigma(x)$ if and only if $(\mathbf{1}\lambda - x)^{-1}$ exists and is bounded in $\mathscr{A}$. This happens if and only if $(\tilde{\mathbf{1}}_\alpha\lambda - \tilde{\phi}_\alpha(x))^{-1}$, where $\tilde{\mathbf{1}}_\alpha$ is the unit element in $\tilde{\mathscr{A}}_\alpha$, exists for some $\alpha \in I$. But that means that $\lambda \in \sigma_\alpha(\tilde{\phi}_\alpha(x))$ for some $\alpha \in I$. On the other hand $\lambda = 0 \in \sigma(x)$ if and only if x is not invertible in $\mathscr{A}$. This happens if and only if $\tilde{\phi}_\alpha(x)$ is not invertible in $\tilde{\mathscr{A}}_\alpha$ for some $\alpha \in I$, that is, if $0 \in \sigma_\alpha(\tilde{\phi}_\alpha(x))$ for some $\alpha \in I$.

2. The first equality follows from the previous demonstration. The second from the known formula for the spectral radius in Banach algebra's theory [Bratteli]. ■

Definition 4. An involution on an algebra $\mathscr{A}$ is defined as a conjugate antiisomorphism of period 2, that is, as an application

$$*: \quad \mathscr{A} \to \mathscr{A}$$

$$x \to x^* \tag{12}$$

that satisfies

$$(x + y)^* = x^* + y^* \tag{13}$$

$$(\lambda x)^* = \bar{\lambda} x^* \tag{14}$$

$$(xy)^* = y^* x^* \tag{15}$$

$$(x^*)^* = x \tag{16}$$

for every pair $x, y \in \mathscr{A}$ and every $\lambda \in \mathbb{C}$. An algebra in which is defined on an involution is called an involutive algebra or a *-algebra.

Definition 5. A symmetric element of a topological *-algebra with unit is called Hermitian if and only if its spectrum is contained in the field of the real numbers. If every symmetric element is Hermitian, then one says that the involution is Hermitian.

Definition 6. A locally convex *-algebra with unit is called regular if for every element $x \in \mathscr{A}$ it is verified that $(\mathbf{1} + x^*x)$ is invertible. If, moreover, $(\mathbf{1} + x^*x)^{-1}$ is bounded, then the algebra is called symmetric.

There exists an important case in which regularity implies symmetry.

Proposition 6 [Allan] Let $\mathscr{A}$ be a totally convex *-algebra with unit and continuous inversion. If for every element $x \in \mathscr{A}$ one has that $(\mathbf{1} + x^*x)$ is invertible, i.e., $\mathscr{A}$ is regular, then $\mathscr{A}$ is symmetric.

Definition 7. A submultiplicative seminorm p defined on a *-algebra $\mathscr{A}$ is regular if it satisfies

$$p(x^*x) = p(x)^2 \tag{17}$$

for every $x \in \mathscr{A}$.

Definition 8. A complete topological *-algebra in which there exists a basis of continuous seminorms composed entirely by regular seminorms is called a b*-algebra.

Proposition 7 [Allan]. Every b*-algebra with unit is regular.

Remark 2. Since it is locally convex and has continuous inversion, it follows from Proposition 6 that every b*-algebra is symmetric.

Of course, as locally multiplicatively convex algebras are related to normed algebras, in a similar way, b*-algebras are related to C*-algebras.

Proposition 8. A topological *-algebra is a locally multiplicatively convex *-algebra with a basis of continuous regular seminorms if and only if it is isomorphic to a *-subalgebra of the direct product of a collection of normed regular algebras. The algebra is a b*-algebra if and only if it is the Hausdorff projective limit of a family of C*-algebras.

The proof of this proposition is essentially the same of that of Theorem 1.

Another connection between b*-algebras and C*-algebras is given by the following theorem, closely related to the fact that every C*-algebra is realized through the GNS construction as an algebra of bounded operators on a Hilbert space.

Theorem 3. Let $\mathcal{A}$ be a barreled b*-algebra with unit and let $\mathcal{A}_0$ be the collection of bounded elements of $\mathcal{A}$. Then $\mathcal{A} = \mathcal{A}_0$ if and only if $\mathcal{A}$ is a C*-algebra with unit.

Proof. We will show that if $\mathcal{A} = \mathcal{A}_0$, then $\mathcal{A}$ is a b*-algebra with unit. The demonstration of the converse is trivial, so we will omit it.

Let $\{p_\alpha\}_{\alpha \in I}$ be a basis of continuous regular seminorms of $\mathcal{A}$ and let B_0 be the set given by

$$B_0 = \{x \in \mathcal{A}\colon p_\alpha(x) \leq 1, \forall \alpha \in I\} \tag{18}$$

Observe that B_0 is a close, absolutely convex bounded subset of $\mathcal{A}_0$, that $B_0^2 \subset B_0$, that $\mathbf{1} \in B_0$, and that $B_0^* = B_0$. Moreover, one has that any subset with these properties is included in B_0. Let

$$\mathcal{A}(B_0) = \{\lambda x\colon \lambda \in \mathbb{C}, x \in B_0\} \tag{19}$$

Clearly, one has that $\mathcal{A}(B_0)$ is a *-subalgebra of $\mathcal{A}$ that contains all normal elements of $\mathcal{A}_0$.

In this case, $\mathcal{A} = \mathcal{A}_0 = \mathcal{A}(B_0)$, so B_0 is absorbing in $\mathcal{A}$. Because $\mathcal{A}$ is barreled, B_0 is a neighborhood of zero of $\mathcal{A}$. But B_0 is bounded, so $\mathcal{A}$ is a normed algebra.

On the other hand, it can be proved [Allan] that the Minkowski functional associated with B_0,

$$p_0\colon \quad \mathcal{A}(B_0) \to \mathbb{R}_+$$
$$x \to p_0(x) = \mathit{inf}\{\lambda > 0\colon (1/\lambda)x \in B_0\} \tag{20}$$

defines in $\mathcal{A}(B_0)$ a norm that makes it a C*-algebra with unit.

The completeness of $\mathcal{A}$ and the fact that all complete norms on a C*-algebra are equivalent end the demonstration. ∎

Corollary 1. Let $\mathcal{A}$ be a nuclear barreled b*-algebra with unit and let $\mathcal{A}_0$ be the collection of bounded elements of $\mathcal{A}$. Then $\mathcal{A} = \mathcal{A}_0$ if and only if $\mathcal{A}$ is a finite-dimensional C*-algebra with unit.

Proof. The proof is trivial, considering that every nuclear normed space is finite dimensional [Treves]. ∎

In the following sections we will say that an algebra $\mathcal{A}$ is a *characteristic algebra* if it is a nuclear barreled b*-algebra with unit.

3. OBSERVABLES AND STATES: THE EXTREMAL DECOMPOSITION THEOREM

Definition 9. Let $\mathcal{A}$ be a characteristic algebra. An element $x \in \mathcal{A}$ will be called an observable if and only if its spectrum is real. We will denote the set of observables by $\mathcal{A}_S$.

Proposition 9. Let $\mathcal{A}$ be a characteristic algebra. An element is an observable if and only if it is a symmetric element.

Proof. Let $x \in \mathcal{A}$ be an observable. Then, its spectrum is contained in the real line. So, for every $\alpha \in I$ we have that $\sigma_\alpha(\hat{\phi}_\alpha(x)) \subset \mathbb{R}$ and then $\hat{\phi}_\alpha(x) = \hat{\phi}_\alpha(x)^* = \hat{\phi}_\alpha(x^*)$, and $x^* = x$. The converse follows from Proposition 7. ■

It must be evident now that $\mathcal{A}_S$ is a real, closed subspace of $\mathcal{A}$, and so a real, complete nuclear barreled space.

Definition 10. An observable $x \in \mathcal{A}_S$ is positive if it is the square of another observable, that is, if there exists $y \in \mathcal{A}_S$ such that $x = y^2$. We will denote the set of positive observables by $\mathcal{A}_{S+}$.

Proposition 10. Let $\mathcal{A}$ be a characteristic algebra, $\mathcal{A}_S$ the collection of observables, and $\mathcal{A}_{S+}$ the set of positive elements of $\mathcal{A}_S$. An observable $x \in \mathcal{A}_S$ is positive if and only if $\hat{\phi}_\alpha(x)$ is positive for every $\alpha \in I$.

Proof. If $x \in \mathcal{A}_S$ is positive, one has that there exists an element $y \in \mathcal{A}_S$ such that $x = y^2$. With this, $\hat{\phi}_\alpha(x) = \hat{\phi}_\alpha(y^2)$ for every $\alpha \in I$. Because each $\hat{\phi}_\alpha$ is an homomorphism, one has that $\hat{\phi}_\alpha(x) = \hat{\phi}_\alpha(y)^2$, so $\hat{\phi}_\alpha(x)$ is positive. Reciprocally, suppose that $\hat{\phi}_\alpha(x) = \hat{\phi}_\alpha(y^2)$ is positive for every $\alpha \in I$ and consider the element $y = (\sqrt{\hat{\phi}_\alpha(x)})_{\alpha \in I}$. It is evident that this element is an element of $\mathcal{A}_S$ and that it satisfies $x = y^2$. ■

Theorem 4. $\mathcal{A}_{S+}$ is a reproductive strict convex cone closed in $\mathcal{A}_S$.

Proof. The demonstration of the fact that $\mathcal{A}_{S+}$ is a strict cone is elemental, like that of its closure in $\mathcal{A}_S$. The convexity of $\mathcal{A}_{S+}$ follows from the last two propositions. Finally, the fact that $\mathcal{A}_{S+}$ is a reproductive cone follows from the existence of an internal point in $\mathcal{A}$, i.e., the unit element, and from Proposition 10. ■

Theorem 5. An observable is positive if and only if its spectrum is contained in $\mathbb{R}^+$.

Proof. The demonstration of this proposition follows from the fact that in every C*-algebra a symmetric element is positive if and only if its spectrum is in $\mathbb{R}^+$ and from Proposition 8. ■

Definition 11. A functional over $\mathcal{A}_S$ is positive if it takes nonnegative values on the cone of positive observables. We will denote the set of positive functionals by $\mathcal{A}'_{S+}$, so we have

$$\mathcal{A}'_{S+} = \{\rho \in \mathcal{A}'_S\colon \rho(x) \geq 0,\ \forall x \in \mathcal{A}_{S+}\} \tag{21}$$

Proposition 11 [Treves]. $\mathcal{A}'_{S+}$ is a convex strict reproductive cone (weakly) closed in $\mathcal{A}'_S$.

The following theorem is an extension of the known Krein–Millman lemma.

Theorem 6. Let Γ be a subcone of $\mathcal{A}'_{S+}$. Then Γ is generated by the set $\partial\Gamma$ of its extremal elements

$$\Gamma = \overline{C^0}(\partial\Gamma) \tag{22}$$

where by $\overline{C^0}$ we denote the weak closure of the convex hull of the set.

Proof. For the demonstration of this theorem we will use again the characterization of $\mathcal{A}_S$ as a projective limit. In the first place we have that

$$\Gamma = \bigcup_{\alpha \in I} \Gamma \cap \mathcal{U}^0_\alpha \tag{23}$$

where $\mathcal{U}^0_\alpha$ we denote the polar set associated with the seminorm p_α, that is,

$$\mathcal{U}^0_\alpha = \{\rho \in \mathcal{A}'_S\colon |\rho(x)| \leq 1,\ \forall x \in \mathcal{U}_\alpha\} \tag{24}$$

This follows from the fact that, because $\mathcal{A}_S$ is barreled, every bounded subset of $\mathcal{A}'_S$ is equicontinuous. By identical arguments one has that

$$\partial\Gamma = \bigcup_{\alpha \in I} \partial\Gamma \cap \mathcal{U}^0_\alpha = \bigcup_{\alpha \in I} \partial(\Gamma \cap \mathcal{U}^0_\alpha) \tag{25}$$

On the other hand, $\Gamma \cap \mathcal{U}^0_\alpha$ is a weakly compact subset of $\mathcal{A}'_{S+}$, and so we are under the hypothesis of the Krein–Millman lemma. So,

$$\Gamma \cap \mathcal{U}^0_\alpha = \overline{C^0}(\partial(\Gamma \cap \mathcal{U}^0_\alpha)) \tag{26}$$

The result follows from the three identities. ■

Definition 12. We define the space of states as the section of normal positive continuous functionals on $\mathcal{A}_S$, that is,

$$N(\mathcal{A}'_{S+}) = \{\rho \in \mathcal{A}'_{S+}\colon \rho(\mathbf{1}) = 1\} \tag{27}$$

Of course, this is a weakly closed convex subset of $\mathscr{A}'_{S+}$. The extremal elements of this convex set, which we denote by $\partial N(\mathscr{A}'_{S+})$, are simply the normal elements of $\partial\mathscr{A}'_{S+}$, and we will call them *pure states*.

It is easy to show that $\partial N(\mathscr{A}'_{S+})$ is the collection of indecomposable states, that is, states that do not admit a convex decomposition.

As in the normed case, the extremal decomposition theorem has an integral expression.

Theorem 7 [Hegerfeldt]. Let Γ be as in the previous theorem and let us consider the set $\partial\Gamma$ of extremal elements of Γ. For every $\rho \in \Gamma$ there exists a Radon measure μ over $\partial\Gamma$ such that

$$\rho = \int_{\partial\Gamma} \xi \, d\mu(\xi) \tag{28}$$

Remark 3. For the demonstration of the last theorem, nuclearity is an essential assumption.

Remark 4. Observe that, in particular, we can take $\Gamma = N(\mathscr{A}'_{S+})$.

4. FUNCTIONAL REPRESENTATION THEORY

Definition 13. We will say that a subset $\hat{\mathscr{A}}$ of $\mathscr{A}_S$ is a system of observables if and only if it is a closed subalgebra with unit of $\mathscr{A}$ and is barreled provided with the inherited topology. If a system of observables $\hat{\mathscr{A}}$ is a maximal subalgebra of $\mathscr{A}$, we will say that it is a complete system of observables.

Observe that, as a closed subalgebra of $\mathscr{A}$, every system of observables $\hat{\mathscr{A}}$ is a real, commutative, complete, nuclear, locally multiplicatively convex algebra with unit in which, in addition, we have

$$\hat{p}_\alpha(x^2) = (\hat{p}_\alpha(x))^2 \tag{29}$$

for every $x \in \hat{\mathscr{A}}$, where $\{\hat{p}_\alpha\}_{\alpha\in I}$ is the basis of submultiplicative seminorms of $\mathscr{A}$ restricted to $\hat{\mathscr{A}}$.

By $\hat{\mathscr{A}}_\alpha$ we will represent the Banach algebra which is the completion of $\hat{\mathscr{A}}/Ker(\hat{p}_\alpha)$ with respect the topology induced by $\hat{p}_\alpha$ and with $\hat{\mathscr{A}}_+$ the cone associated with the ordering inherited by $\hat{\mathscr{A}}$ from $\mathscr{A}_S$.

Definition 14. Let $\hat{\mathscr{A}}$ be a system of observables. Let $\hat{\mathcal{M}}$ be the family of maximal closed ideals of $\hat{\mathscr{A}}$. We will say that $\hat{\mathcal{M}}$ is the spectral space associated with $\hat{\mathscr{A}}$.

It can be shown [Michael] that there exists a one-to-one correspondence between $\hat{\mathcal{M}}$ and the collection of the nontrivial multiplicative functionals on $\hat{\mathscr{A}}$, that is, that functionals $\hat{\rho} \in \hat{\mathscr{A}}'$ that satisfy

$$\hat{\rho}(xy) = \hat{\rho}(x)\hat{\rho}(y) \tag{30}$$

for every pair of elements $x, y \in \hat{\mathcal{A}}$. We will denote this collection by $\partial N(\hat{\mathcal{A}}'_+)$. According to these correspondence, to each $\mathcal{T}_\rho \in \hat{\mathcal{M}}$ there is associated an element $\hat{\rho} \in \partial N(\hat{\mathcal{A}}'_+)$ such that

$$\mathcal{T}_\rho = Ker(\hat{\rho}) \tag{31}$$

It can be shown that $\hat{\mathcal{M}}$ provided with the finer topology that makes this correspondence a continuous application, observing in $\partial N(\hat{\mathcal{A}}'_+)$ the weak topology, is a convex Hausdorff space. So, we can identify $\hat{\mathcal{M}}$ with $\partial N(\hat{\mathcal{A}}'_+)$.

Proposition 12. Every compact subset of $\hat{\mathcal{M}}$ is equicontinuous.

Proof. The demonstration of this proposition is immediate upon observing that for a subset of the dual space of a barreled space is equivalent to saying that this subset is equicontinuous to saying that it is relatively compact in the weak topology [Treves]. ■

Proposition 13 [Michael]. Let $\mathcal{U}_\alpha^0$ be the polar set associated with the seminorm $\hat{p}_\alpha$, for every $\alpha \in I$. One has the following:

1. $\hat{\mathcal{M}} = \cup_{\alpha \in I}(\hat{\mathcal{M}} \cap \mathcal{U}_\alpha^0)$.
2. $\hat{\mathcal{M}} \cap \mathcal{U}_\alpha^0$ is a compact set for every $\alpha \in I$.
3. Each map that assigns to each element $\hat{\rho} \in \hat{\mathcal{M}} \cap \mathcal{U}_\alpha^0$ the element $\hat{\rho}_\alpha \in \hat{\mathcal{M}}_\alpha$, where by $\hat{\mathcal{M}}_\alpha$ we denote, of course, the collection of multiplicative functionals on the Banach algebra $\hat{\mathcal{A}}_\alpha$, according to the expression

$$\hat{\rho}(x) = \hat{\rho}_\alpha(\hat{\phi}_\alpha(x)) \tag{32}$$

for every $x \in \hat{\mathcal{A}}$, is a continuous one-to-one homeomorphism on $\hat{\mathcal{M}}_\alpha$.

Proposition 14 [Michael]. Let $\hat{\rho}$ be a multiplicative functional on $\hat{\mathcal{A}}$. Then, for every $x \in \hat{\mathcal{A}}$, $\hat{\rho}(x) \in \sigma(x)$.

Theorem 8. Let $\hat{\mathcal{A}}_x$ be the system of observables generated by an element $x \in \mathcal{A}_S$, that is, the completion of the polynomial algebra in x with real coefficients with respect to the topology induced by $\mathcal{A}_S$. Then, the spectral space associated with $\hat{\mathcal{A}}_x$, $\hat{\mathcal{M}}_x$, is homeomorphic to the spectrum of the observable x.

Proof. Cotlar [Cotlar71] demonstrates an analogous result in the context of commutative C-algebras. So, for every $\alpha \in I$ such that $x \notin Ker(\hat{p}_\alpha)$, one has that $\hat{\mathcal{M}}_{x\alpha}$, the spectral space associated with $\hat{\mathcal{A}}_{x\alpha}$, is homeomorphic to the spectrum of $\hat{\phi}_\alpha(x)$. From Proposition 8 it follows that $\hat{\mathcal{M}}_x \cap \mathcal{U}_\alpha^0$ is homeomorphic to $\sigma(\hat{\phi}_\alpha(x))$. With this and Theorem 2 we end the demonstration. ■

Theorem 9. Every system of observables $\hat{\mathcal{A}}$ is isomorphic to the space of continuous functions defined on the spectral space of $\hat{\mathcal{A}}$, $C(\hat{\mathcal{M}})$, provided with the topology of uniform convergence on the collection of compact subsets of $\hat{\mathcal{M}}$. Moreover, each element that is never -1 has its inverse in it.

Proof. Consider the Gel'fand transform

$$\varphi: \quad \hat{\mathcal{A}} \to C(\hat{\mathcal{M}})$$

$$x \to \varphi(x) \tag{33}$$

given by the expression

$$(\varphi(x))(\hat{\rho}) = \hat{\rho}(x) \tag{34}$$

for every $\hat{\rho} \in \hat{\mathcal{M}}$ and consider the image of $\hat{\mathcal{A}}$ under φ, $\varphi(\hat{\mathcal{A}})$.

It is evident that φ is a homomorphism from $\hat{\mathcal{A}}$ on $\varphi(\hat{\mathcal{A}})$ and, because $Ker(\varphi) = \{0\}$, it is an algebraic isomorphism. Moreover, if $(\varphi(x))(\hat{\rho}) \neq -1$ for every $x \in \hat{\mathcal{A}}$ one has that x is invertible [Michael] and $\varphi(x^{-1}) = \varphi^{-1}(x)$, so the second part of the theorem is proved.

Let us check that $\varphi(\hat{\mathcal{A}}) = C(\hat{\mathcal{M}})$. Let $z \in C(\hat{\mathcal{M}})$ and for every $\alpha \in I$ let us define $z_\alpha = z\big|_{\hat{\mathcal{M}} \cap \mathcal{U}_\alpha^0}$. We associate to each z_α the element $x_\alpha \in C(\hat{\mathcal{M}}_\alpha)$ by virtue of the homeomorphism that exists between $\hat{\mathcal{M}} \cap \mathcal{U}_\alpha^0$ and $\hat{\mathcal{M}}_\alpha$. As it is known that $\hat{\mathcal{A}}_\alpha$ (see the classical Gel'fand transform theorem [Bratteli]) is isomorphic to $C(\hat{\mathcal{M}}_\alpha)$, we can identify x_α with the corresponding element in $\hat{\mathcal{A}}_\alpha$. It is easy to see that, using the notation introduced in Theorem 1, $\widetilde{\phi_{\alpha\beta}}(x_\alpha) = x_\beta$ if $\alpha \geq \beta$. With this and the fact that $\hat{\mathcal{A}}$ is the Hausdorff projective limit of the collection $\{\hat{\mathcal{A}}_\alpha\}_{\alpha \in I}$ we have, finally, that $x = (x_\alpha)_{\alpha \in I} \in \hat{\mathcal{A}}$ satisfies $\varphi(x) = z$.

Now, let us show that the isomorphism is a topological one. Given an equicontinuous subset $\mathcal{K} \subset \hat{\mathcal{M}}$, for a zero neighborhood $\mathcal{U}$ of $\hat{\mathcal{A}}$ we have that $\mathcal{K} \subset \mathcal{U}^0 \cap \hat{\mathcal{M}}$; considering that $\mathcal{U}^0$ is weakly compact in $\hat{\mathcal{A}}'$ and that $\hat{\mathcal{M}}$ is weakly closed in $\hat{\mathcal{A}}'$, it follows that $\mathcal{U}^0 \cap \hat{\mathcal{M}}$ is a compact subset of $\hat{\mathcal{M}}$. With this fact it is clear that the topology of uniform convergence on the equicontinuous subsets of $\hat{\mathcal{M}}$ and the topology of uniform convergence on the compact equicontinuous subsets of $\hat{\mathcal{M}}$ are equivalent. Considering Proposition 12, it is clear that both coincide with the topology of uniform convergence on the compact subsets of $\hat{\mathcal{M}}$ and that, identifying $\hat{\mathcal{A}}$ with $C(\hat{\mathcal{M}})$, the original topology in $\hat{\mathcal{A}}$ is finer.

So, noting that the topology in $\hat{\mathcal{A}}_\alpha$ is the one of the uniform convergence, one has that for every $x \in \hat{\mathcal{A}}$ there exists a real number $\lambda_\alpha > 0$ such that

$$\tilde{p}_\alpha(x) \leq \lambda_\alpha \, sup\{|\hat{\rho}_\alpha(\phi_\alpha(x))|: \hat{\rho}_\alpha \in \hat{\mathcal{M}}_\alpha\}$$
$$\leq \lambda_\alpha \, sup\{|\hat{\rho}(x)|: \hat{\rho} \in \hat{\mathcal{M}} \cap \mathcal{U}_\alpha^0\} \tag{35}$$

This last expression and the fact that $\hat{\mathcal{M}} \cap \mathcal{U}_\alpha^0$ is an equicontinuous subset of $\hat{\mathcal{M}}$ end the demonstration. ■

Of course, there exists in this case a decomposition theorem analogous to Theorems 6 and 7.

Theorem 10. Let $\hat{\Gamma}$ be a convex subcone of $\hat{\mathcal{A}}'_+$. Then:

1. $\hat{\Gamma}$ is generated by the set $\partial\hat{\Gamma}$ of its extremal elements, that is,

$$\hat{\Gamma} = \overline{C^0}(\partial\hat{\Gamma}) \tag{36}$$

2. For every $\hat{\rho} \in \hat{\Gamma}$ there exists a Radon measure μ on $\partial\hat{\Gamma}$, the set of extremal elements of $\hat{\Gamma}$, such that

$$\hat{\rho} = \int_{\partial\hat{\Gamma}} \hat{\xi} \, d\mu(\hat{\xi}) \tag{37}$$

Remark 5. The commutativity of the product on $\hat{\mathcal{A}}$ guarantees that the face $\hat{\Gamma}(\hat{\rho})$ is a lattice with respect to the order induced by $\hat{\Gamma}$. This fact is essential to demonstrate that the measure μ is unique.

5. THE GNS REPRESENTATION THEOREM

Definition 15. Let $\mathcal{A}$ be a *-algebra. A linear functional ρ defined on $\mathcal{A}$ is positive if and only if $\rho(x^*x) \geq 0$ for every $x \in \mathcal{A}$. We will denote the collection of all positive functionals by $\mathcal{A}'_+$.

Proposition 15 [Allan]. Let $\mathcal{A}$ be a *-algebra and ρ a positive functional on $\mathcal{A}$.

1. $\rho(x^*) = \overline{\rho(x)}$ for every $x \in \mathcal{A}$.
2. The Cauchy–Schwarz inequality is satisfied, that is, for every pair of elements $x, y \in \mathcal{A}$

$$|\rho(y^*x)| \leq \rho(y^*y)^{1/2}\rho(x^*x)^{1/2} \tag{38}$$

3. If $\mathcal{A}$ is a Banach *-algebra with unit, then ρ is continuous and

$$\|\rho\| = \sup_{\|x\|\leq 1} |\rho(x)| = \rho(\mathbf{1}) \tag{39}$$

where, of course, **1** represents the unit in $\mathcal{A}$

Proposition 16. Let $\mathscr{A}$ be a *-algebra. There exists a bijection between $\mathscr{A}'_{S+}$ and $\mathscr{A}'_{+}$ such that every extremal element of $\mathscr{A}'_{S+}$ is in correspondence with an extremal element of $\mathscr{A}'_{+}$.

Proof. It is clear that the restriction of every element of $\mathscr{A}'_{+}$ defines a positive functional on $\mathscr{A}_S$ and that if the element is extremal, then its restriction is extremal.

So, we must demonstrate that every functional $\rho \in \mathscr{A}'_{S+}$ can be extended under the same conditions to an element of $\mathscr{A}'_{+}$.

It is easy to see that if we define on $\mathscr{A}$ the functional given by

$$\tilde{\rho}(x) = \rho(Re(x)) - i\rho(Im(x)) \tag{40}$$

where

$$Re(x) = \frac{x + x^*}{2} \tag{41}$$

and

$$Im(x) = \frac{x - x^*}{2i} \tag{42}$$

we have a positive form on $\mathscr{A}$.

The indecomposability of $\tilde{\rho}$ follows from that of ρ and the fact that $Re(\tilde{\rho}(x)) = \rho(Re(x))$ and $Im(\tilde{\rho}(x)) = -i\rho(Im(x))$. ■

Proposition 17 [Belanger]. Let $\mathscr{E}$ be a complete locally convex Hausdorff space. There exists a bijection between the set of Hilbert subspaces of $\mathscr{E}$ and the family of positive operators defined from $\mathscr{E}^{\times}$ (the antidual space of $\mathscr{E}$) into $\mathscr{E}$; we will denote this family by $\mathscr{L}^{+}(\mathscr{E}^{\times}, \mathscr{E})$.

Proof.

1. *Existence:* Let $N = \{\rho \in \mathscr{E}^{\times}: \langle H\rho, \rho\rangle = 0\}$, $H \in \mathscr{L}^{+}(\mathscr{E}^{\times}, \mathscr{E})$. Consider the quotient space $\mathscr{E}^{\times}/N$ and let us denote by Ω the pre-Hilbertian space $\mathscr{E}^{\times}/N$ provided with the inner product derived from the nonnegative sesquilinear form that defines the operator H, i.e.,

$$h: \quad \mathscr{E}^{\times} \times \mathscr{E}^{\times} \to \mathbb{C}$$

$$(\rho, \xi) \to h(\rho, \xi) = \langle H\rho, \xi\rangle \tag{43}$$

quotient modulo N. Finally, let us denote by ϕ the canonical mapping from $\mathscr{E}^{\times}$ on $\mathscr{E}^{\times}/N$.

From the Cauchy–Schwarz inequality

$$\left|\langle H\rho, \xi\rangle\right| \leq \langle H\rho, \rho\rangle^{1/2}\langle H\xi, \xi\rangle^{1/2} \tag{44}$$

it is easy to derive that $N = \{\rho \in \mathscr{E}^\times: H\rho = 0\}$, so the linear mapping

$$\begin{aligned} \hbar: \quad & \Omega \to \mathscr{E}^\times \\ \phi(\rho) \to & \hbar\phi(\rho) = H\rho \end{aligned} \tag{45}$$

is continuous and can be uniquely extended to the completion of Ω. Let us denote by $\tilde{H}$ the corresponding extended map.

Clearly, $\mathscr{H} = Im(\tilde{H})$ with the Hilbertian norm that makes $\tilde{H}$ an isometry is a Hilbert subspace of $\mathscr{E}$ with H as reproducing operator.

2. *Uniqueness:* It will be sufficient to see that $\mathscr{H}$ is determined by H. From the equations

$$\langle jx, \rho\rangle = \langle x, j^\times\rho\rangle = b(x, j^\times\rho) \tag{46}$$

and

$$H = jj^\times \tag{47}$$

where j is the injection of $\mathscr{H}$ in $\mathscr{E}$, is clear that the subspace $j^\times(\mathscr{E}^\times)$ is dense in $\mathscr{H}$.

The unitary ball B of $\mathscr{H}$ is weakly compact in $\mathscr{E}$, so is weakly closed in $\mathscr{E}$. Since B is a convex set, B is closed in $\mathscr{E}$.

One has, thus, that B is the closure in $\mathscr{E}$ of the set $\{H\rho: \langle H\rho, \rho\rangle^{1/2} \leq 1\}$; this shows that $\mathscr{H}$ is defined by H. Moreover, it can be proved [Belanger] that given $x \in \mathscr{H}$,

$$\|x\| = \left\{sup \frac{|\langle x, \rho\rangle|}{\langle H\rho, \rho\rangle^{1/2}} : \rho \in \mathscr{E}^\times, \langle H\rho, \rho\rangle > 0\right\} \tag{48}$$

This ends the demonstration. ■

Remark 6. Every Hilbert subspace $\mathscr{H} \overset{}{\underset{j}{\hookrightarrow}} \mathscr{E}$ contains a privileged dense subspace, $\mathscr{D}_{\mathscr{H}} = j^\times(\mathscr{E}^\times)$.

Remark 7. The map $j^\times$ is one to one if and only if $\mathscr{H}$ is dense in $\mathscr{E}$. In this case the triplet $(\mathscr{E}^\times, \mathscr{H}, \mathscr{E})$ is called a Gel'fand triplet [Gel'fand64a].

Consider, now, a positive continuous form ρ on a topological *-algebra $\mathscr{A}$ and the sesquilinear form defined by

$$\begin{aligned} h_\rho: \quad & \mathscr{A} \times \mathscr{A} \to \mathbb{C} \\ (x, y) \to & h_\rho(x, y) = \rho(y^* x) \end{aligned} \tag{49}$$

From now on we will identify the dual space of $\mathscr{A}$ with its corresponding antidual space observing the relation

$$\xi(x) = \langle x^*, \xi\rangle \tag{50}$$

for every $x \in \mathscr{A}$ and every $\xi \in \mathscr{A}'$.

With this notation we have that

$$h_\rho(x, y) = \langle x^* y, \rho \rangle \tag{51}$$

for every pair of elements $x, y \in \mathcal{A}$.

The action of the algebra $\mathcal{A}$ on its (anti)dual space is defined naturally by

$$\langle y, x\xi \rangle = \langle x^* y, \xi \rangle \tag{52}$$

for every $\xi \in \mathcal{A}'$ and every $x, y \in \mathcal{A}$. This identity defines, of course, a representation of $\mathcal{A}$ on its dual. It can be proved that this representation is separately continuous if $\mathcal{A}'$ is provided with the weak topology, or with the strong topology if, as in our case, the algebra is barreled.

Observing that ρ is positive, the kernel defined by equation (49) is associated with a reproducing operator:

$$H_\rho: \quad \mathcal{A} \to \mathcal{A}'$$

$$x \to H_\rho x = x\rho \tag{53}$$

It must be obvious that the kernel h_ρ and the operator H_ρ are invariant under the action of $\mathcal{A}$, that is,

$$h_\rho(xz, y) = h_\rho(z, x^* y) \tag{54}$$

for every $x, y, z \in \mathcal{A}$, or

$$xH_\rho y = H_\rho xy \tag{55}$$

for every pair $x, y \in \mathcal{A}$.

So, consider the Hilbert subspace $\mathcal{H}_\rho \overset{j}{\hookrightarrow} \mathcal{A}'$ associated with the reproducing operator H_ρ. The subspace $\mathcal{D}_{\mathcal{H}_\rho} \subset \mathcal{H}_\phi$ given by

$$\mathcal{D}_{\mathcal{H}_\rho} = j^\times \mathcal{A} = \{x\rho: x \in \mathcal{A}\} \tag{56}$$

is, clearly, dense in $\mathcal{H}_\rho$ and invariant under the left multiplication by elements of $\mathcal{A}$.

With these definitions is clear that the inner product in $\mathcal{H}_\rho$ follows from the extension by continuity of the form

$$b: \quad \mathcal{D}_{\mathcal{H}_\rho} \times \mathcal{D}_{\mathcal{H}_\rho} \to \mathbb{C}$$

$$(x\rho, y\rho) \to b(x\rho, y\rho) = h_\rho(x, y) \tag{57}$$

to all $\mathcal{H}_\rho$.

We will denote by π_ρ the representation of $\mathcal{A}$ that is obtained as a restriction to $\mathcal{D}_{\mathcal{H}_\rho}$ of the representation of $\mathcal{A}'$. So, we have

$$\pi_\rho(x)\varphi = x\varphi \tag{58}$$

for every $x \in \mathcal{A}$ and every $\varphi \in \mathcal{D}_{\mathcal{H}_\rho}$.

Finally, π_ρ is a *-representation of $\mathcal{A}$ in $\mathcal{H}_\rho$, in general, by unbounded operators with common invariant domain $\mathcal{D}_{\mathcal{H}_\rho}$.

Definition 16. The triplet $(\pi_\rho, \mathcal{H}_\rho, \mathcal{D}_{\mathcal{H}_\rho})$ is called the GNS representation of $\mathcal{A}$.

Finally, we have the following extension of the GNS representation theorem.

Theorem 11. Let $\mathcal{A}$ be a nuclear barreled b*-algebra with unit $\rho \in \mathcal{A}'_+$ a positive form on $\mathcal{A}$. There exists a unique (up to unitary equivalence) cyclic *-representation, the GNS representation of $\mathcal{A}$, π_ρ, on a Hilbert space $\mathcal{H}_\rho$ such that

$$\rho(x) = b(\pi_\rho(x)\phi, \phi) \tag{59}$$

for every $x \in \mathcal{A}$, where b is the inner product in $\mathcal{H}_\rho$ and ϕ the cyclic vector in $\mathcal{H}_\rho$. Moreover, this representation is essentially self-adjoint and the invariant common domain $\mathcal{D}_{\mathcal{H}_\rho}$ of the operators representing $\mathcal{A}$ can be topologized to conform a rigged Hilbert space. Observing this topology, all the operators representing $\mathcal{A}$ are continuous on $\mathcal{D}_{\mathcal{H}_\rho}$.

Proof. The uniqueness of the GNS representation follows from the fact that the expression

$$\rho(x) = h_\rho(x, \mathbf{1}) \tag{60}$$

defines a homeomorphism between the family of positive functionals on $\mathcal{A}$ and the collection of Hilbert subspaces of $\mathcal{A}$ (see Theorem 17). The fact that $\rho\mathbf{1}$ is a cyclic vector for the representation is evident. Finally, essentially self-adjointness follows from the symmetry of the algebra. On the other hand, as $\mathcal{D}_{\mathcal{H}_\rho}$ is isomorphic to the quotient space $\mathcal{A}/\mathcal{K}_\rho$, where $\mathcal{K}_\rho$ is the kernel of the reproducing operator H_ρ (see Theorem 17) and $\mathcal{K}_\rho$ is closed in $\mathcal{A}$, we have that $\mathcal{D}_{\mathcal{H}_\rho}$ is complete, nuclear, and barreled. The form b given by the expression (57) is separately continuous (thus continuous, since $\mathcal{D}_{\mathcal{H}_\rho}$ is barreled) on $\mathcal{D}_{\mathcal{H}_\rho} \times \mathcal{D}_{\mathcal{H}_\rho}$, and we have that $\mathcal{D}_{\mathcal{H}_\rho}$ is a nuclear space rigged by $\mathcal{H}_\rho$ [Gel'fand64b]. ■

Remark 8. The occurrence of this kind of space in quantum mechanics is not new: see, for example, [Bogolubov] and [Roberts].

6. AN EXAMPLE

Let us finally consider an example which is very familiar from quantum mechanics.

Let Ω be a locally convex space. The tensor algebra over Ω, $\mathcal{T}(\Omega)$, is defined as the locally convex direct sum of the integer tensor powers of this space, i.e.,

$$\mathcal{T}(\Omega) = \mathbb{C} \oplus \Omega \oplus (\Omega \hat{\otimes} \Omega) \oplus (\Omega \hat{\otimes} \Omega \hat{\otimes} \Omega) \oplus \cdots \tag{61}$$

or, compactly,

$$\mathcal{T}(\Omega) = \bigoplus_{n=0}^{\infty} \Omega^{\hat{\otimes} n} \tag{62}$$

where, by definition, $\Omega^{\hat{\otimes} 0} = \mathbb{C}$.

Let us remember that the locally convex direct sum of a family of locally convex spaces is defined as the algebraic direct sum of these spaces provided with the weaker topology that makes each of the corresponding canonical injections a continuous map.

The sum and the scalar product are defined componentwise, i.e., if $x = (x_0, x_1, \ldots, x_n, \ldots) \in \mathcal{T}(\Omega)$, $y = (y_0, y_1, \ldots, y_n, \ldots) \in \mathcal{T}(\Omega)$, and $\lambda \in \mathbb{C}$,

$$x + y = (x_0 + y_0, x_1 + y_1, \ldots, x_n + y_n, \ldots) \tag{63}$$

$$\lambda x = (\lambda x_0, \lambda x_1, \ldots, \lambda x_n, \ldots) \tag{64}$$

On the other hand, the multiplication law is given by

$$xy = \left(x_0 \otimes y_0, x_0 \otimes y_1 + x_1 \otimes y_0, \ldots, \sum_{k=0}^{n} x_k \otimes y_{n-k}, \ldots \right) \tag{65}$$

If a continuous conjugation is defined on the space Ω we can define an involution in $\mathcal{T}(\Omega)$ in the following way: given an element $x_k = x_k^1 \otimes x_k^2 \otimes \cdots \otimes x_n^k \in \Omega^{\otimes k}$, we make

$$\begin{aligned}(x_k)^* &= (x_k^1 \otimes x_k^2 \otimes \cdots \otimes x_k^k)^* \\ &= (x_k^k)^* \otimes (x_k^{k-1})^* \otimes \cdots \otimes (x_k^1)^* \end{aligned} \tag{66}$$

We extend by continuity and linearity the operation to $\Omega^{\hat{\otimes} k}$ and finally we define the involution in $\mathcal{T}(\Omega)$ componentwise, i.e., given $x = (x_1, x_2, \ldots, x_n, \ldots) \in \mathcal{T}(\Omega)$, we have

$$\begin{aligned} x^* &= (x_1, x_2, \ldots, x_n, \ldots)^* \\ &= (x_1^*, x_2^*, \ldots, x_n^*, \ldots) \end{aligned} \tag{67}$$

Obviously, the unit element in $\mathcal{T}(\Omega)$ is the element $\mathbf{1} = (1, 0, \ldots, 0, \ldots)$.

Now, suppose that Ω is a nuclear Fréchet space and let us denote by $\mathcal{T}_n(\Omega)$ the direct sum of the tensor powers of Ω of order less than n, i.e.,

$$\mathcal{T}_n(\Omega) = \mathbb{C} \oplus \Omega \oplus (\Omega \hat{\otimes} \Omega) \oplus \cdots \oplus (\overbrace{\Omega \hat{\otimes} \Omega \hat{\otimes} \cdots \hat{\otimes} \Omega}^{n \text{ times}})$$
$$= \bigoplus_{j=0}^{n} \Omega^{\hat{\otimes} j} \tag{68}$$

Clearly, $\mathcal{T}(\Omega)$ is the strict inductive limit of the collection $\{\mathcal{T}_n(\Omega)\}_{n=0}^{\infty}$, so $\mathcal{T}(\Omega)$ is a nuclear $\mathcal{LF}$*-algebra [Belanger]. With this we have that the algebra is complete, barreled, and, of course, nuclear.

To see that, in fact, $\mathcal{T}(\Omega)$ is a b*-algebra it is sufficient to observe that it can be identified with the Hausdorff projective limit of a collection of C*-algebras; remember that, since Ω is a nuclear metrizable space, there exists a nondecreasing basis $\{p_\alpha\}_{\alpha \in I}$ of continuous seminorms such that each seminorm is Hilbertian [Pietsch]. Let us denote by $\hat{\Omega}_\alpha$ the Hilbert space which is the completion of the quotient space $\Omega / Ker(p_\alpha)$ with respect to the quotient norm $\hat{p}_\alpha = p_\alpha / Ker(p_\alpha)$ and consider the tensor C*-algebras $\mathcal{T}(\hat{\Omega}_\alpha) = \oplus_{n=0}^{\infty} \hat{\Omega}_\alpha^{\otimes_H n}$; it must be evident that $\mathcal{T}(\Omega)$ is the Hausdorff projective limit of the set $\{\mathcal{T}(\hat{\Omega}_\alpha)\}_{\alpha \in I}$ with respect to those mappings that inject each $\mathcal{T}(\hat{\Omega}_\alpha)$ into $\mathcal{T}(\hat{\Omega}_\beta)$ if $\alpha \geq \beta$, where the order in the index set I is the induced by the ordering of the basis of seminorms.

As a particular case, consider that in which $\Omega = \mathcal{S}(\mathbb{R}^4)$, the space of rapidly decreasing and infintely differentiable functions on $\mathbb{R}^4$ or Schwartz's space [Borchers90].

7. CONCLUDING REMARKS

As we have said, an algebraic approach to quantum mechanics is intended to be an extension of the traditional Hilbert space formulation. In this work we have incorporated the abstract counterparts of unbounded operators into the algebra that characterizes the physical system under consideration. The main results of the C*-algebra formalism have been proved for a closely related but a more general kind of algebra. We proved an extremal decomposition theorem which is a result closely related to the quantum mechanical spectral postulate and a generalization of the GNS classical representation theorem that deals with a representation of the characteristic algebra on a rigged Hilbert space. In almost all of our demonstrations an essential point was the fact that every b*-algebra is the Hausdorff projective limit of a collection of C*-algebras. With respect to this, it would be interesting to analyze the connection between the GNS representation we proved for b*-algebras and the GNS representations of the corresponding C*-algebras. We observe that, if this connection is satisfactory, very important concepts like the equivalence of states and definitions like that of KMS states could be easily generalized.

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REFERENCES

[Allan] G. R. Allan, On a class of locally convex algebras, *Proc. Lond. Math. Soc.* **3**(17), 91 (1967).

[Arens] R. Arens, A generalization of normed rings, *Pacific J. Math.* **2** (1952).

[Belanger] A. Bélanger and G. F. Thomas, Positive forms on nuclear *-algebras and their integral representations, *Can. J. Math.* **42**(3), 410 (1990).

[Bogolubov] N. N. Bogolubov, A. A. Logunov, and I. T. Todorov, *Introduction to Axiomatic Quantum Field Theory,* Benjamin/Cummings (1975).

[Borchers72] H. J. Borchers, Algebraic aspects of Wightmann field theory, in *Statistical Mechanics and Field Theory,* Halsted Press, New Yor, (1972).

[Borchers90] H. J. Borchers and J. Yngvason, Positivity of Wightman functionals and the existence of local nets, *Commun. Math. Phys.* **127**, 607 (1990).

[Bratteli] O. Bratteli and D. W. Robinson, *Operator Algebras and Quantum Statistical Mechanics,* Springer, New York (1980).

[Cotlar71] M. Cotlar, *Nociones de Espacios Normados,* Ediciones Previas, Editorial Universitaria de Buenos Aires (1971).

[Cotlar68] M. Cotlar, Equipación con Espacios de Hilbert, Cursos y Seminarios de Matemática Facículo 15, FCEyN, UBA (1968).

[Dixmier] J. Dixmier and P. Malliavin, Factorisations des fonctions et de Vecteurs Indéfiniment Différentiables, *Bull. Soc. Math. France* **102**, 321 (1978).

[Doplicher] S. Doplicher, An algebraic spectrum condition, *Commun. Math. Phys.* **1**, 1 (1965).

[Gel'fand41] I. M. Gel'fand, Normierte Ringe, *Rec. Math. (Math. Sbornik)* **9**, 3 (1941).

[Gel'fand64a] I. M. Gel'fand and G. E. Shilov, *Generalized Functions,* Vols. I–III, Academic Press, New York (1964).

[Gel'fand64b] I. M. Gel'fand and N. Vilenkin, *Generalized Functions,* Vol. IV, Academic Press, New York (1964).

[Grothendieck] A. Grothendieck, *Produits Tensoriels Topologiques et Espaces Nucléaires,* Memoirs of AMS, No. 16 (1955).

[Haag64] R. Haag and D. Kastler, An algebraic approach to quantum field theory, *J. Math. Phys.* **5**, 848 (1964).

[Haag93] R. Haag, *Local Quantum Physics,* Springer-Verlag (1993).

[Hegerfeldt] G. C. Hegerfeldt, Extremal decomposition of Wightman functions and of states on nuclear *-algebras by Choquet theory, *Commun. Math. Phys.* **45**, 133 (1975).

[Jordan] P. Jordan, J. von Neumann, and E. P. Wigner, On an algebraic generalization of the quantum mechanics formulation, *Ann. Math.* **35**, 29 (1934).

[Maurin] K. Maurin, *General Eigenfunction and Unitary Representations of Topological Groups,* Polish Scientific Publishers, Warsaw (1968).

[Michael] E. A. Michael, Locally multiplicatively-convex topological algebras, *Mem. Am. Math. Soc.* **11**, 79 (1960).

[Naimark] M. A. Naimark, *Normed Rings,* P. Noordhoff, Groningen (1972).

[Newmann] J. von Neumann, *Mathematical Foundations of Quantum Mechanics,* Princeton University Press, Princeton, New Jersey (1955).
[Pietsch] A. Pietsch, *Nuclear Locally Convex Spaces,* Springer-Verlag (1972).
[Roberts] J. E. Roberts, Rigged Hilbert spaces in quantum mechanics, *Commun. Math. Phys.* **3**, 98 (1966).
[Ruelle] D. Ruelle, States of physical systems, *Commun. Math. Phys.* **3**, 133 (1966).
[Segal47a] I. E. Segal, Irreducible representations of operator algebras, *Am. Math. Soc.* **53**, 73 (1947).
[Segal47b] I. E. Segal, Postulates for general quantum mechanics, *Ann. Math.* **48**(4), 930 (1947).
[Schwartz57] L. Schwartz, *Théorie des Distributions,* Hermann, Paris (1957).
[Schwartz64] L. Schwartz, Sous-espaces Vectoriels Topologiques et Noyaux Associés, *J. Anal. Math.* **13**, 115 (1964).
[Treves] F. Treves, *Topological Vector Spaces, Distributions and Kernels,* Academic Press, New York (1967).
[Wight62] A. S. Wightman, *Rev. Mod. Phys.* **34**, 845 (1962).
[Wight64] A. S. Wightman and L. Garding, Fields as operator valued distributions in relativistic quantum theory, *Ark. Fys.* **23**(13) (1964).